UNIVERSITY OF PENNSYLVANIA

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DONALD L. McDONOUGH

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IN MATHEMATICS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

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ON THE EXPANSION OF A CERTAIN TYPE OF DETERMINANT

By DONALD L. McDONOUGH

1. In a well known textbook in the theory of invariants[1] the following theorem appears:

"If a_{ix}, where $1 \leqq i \leqq r$, are r distinct (binary) linear forms, and A_i are binary forms of orders α_i, where $\sum_{i=1}^{r} \alpha_i = n - r + 1$, then any binary form f of order n can be expressed in the form

$$f = \sum_{i=1}^{r} a_{ix}^{n-\alpha_i} A_i, \tag{1}$$

and the expression is unique."

It is understood that the linear forms a_{ix} are given and the binary forms A_i are to be determined.

Upon equating coefficients of like products of powers of the variables in the expanded form of (1), $n+1$ linear non-homogeneous equations arise in the coefficients of the binary forms A_i which are to be regarded as unknowns. The determinant of the coefficients of these unknowns has a peculiar form. It has been thought that this determinant could be expanded as a numerical multiple of a product of powers of the differences of the roots of the linear forms. It seems, however, that this has never been proved.

It is the desire of the writer to prove:

Theorem I. *Any determinant that arises in this way can be expressed as a numerical multiple of a product of powers of the resultants of the linear forms taken by pairs.*

Theorem II. *Any such determinant is different from zero, and the numerical multiplier can be expressed in terms of binomial coefficients.*

The truth of both of these theorems is immediately evident in the case where $r = 2$. In this case the determinant in question is merely the resultant of the two

[1] O. E. Glenn's *The Theory of Invariants*, page 120. For a discussion of a somewhat similar problem for $r = 2$ see a paper by the same author, *The Symbolical Theory of Finite Expansions*, Transactions of the American Mathematical Society, vol. 15 (1914), especially pp. 80–82.

forms $(a_{i1}x_1+a_{i2}x_2)^{n-\alpha_i}$, where $\alpha_1+\alpha_2=n+1$. Consequently this determinant is equal to $(a_{11}a_{22}-a_{21}a_{12})^{(n-\alpha_1)(n-\alpha_2)}$. Since the linear forms are distinct, this cannot be zero.

2. For the sake of convenience in notation let the r linear forms be represented in the form $x_1+a_ix_2$, and set $n-\alpha_i=\eta_i$, and $\alpha_i+1=n_i$. Let

$$A_i = \sum_{k=1}^{n_i} p_{ik}x_k.$$

(1) will then take the form

$$\sum_{i=1}^{r}\sum_{j=0}^{\eta_i}\sum_{k=1}^{n_i}\binom{\eta_i}{j}a_i^j p_{ik}x_1^{n-j-k+1}x_2^{j+k-1} = f. \tag{2}$$

The general form of the determinant that arises when we interchange rows and columns is now

$$\begin{vmatrix}
1 & \eta_1 a_1 & \binom{\eta_1}{2}a_1^2 & \cdots & a_1^{\eta_1} & 0 & \cdots & 0 \\
0 & 1 & \eta_1 a_1 & \cdots & \eta_1 a_1^{\eta_1-1} & a_1^{\eta_1} & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1 & \eta_1 a_1 & \binom{\eta_1}{2}a_1^2 & \cdots & \cdots & a_1^{\eta_1} \\
1 & \eta_2 a_2 & \binom{\eta_2}{2}a_2^2 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 1 & \eta_2 a_2 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 & \eta_2 a_2 & \binom{\eta_2}{2}a_2^2 & \cdots & a_2^{\eta_2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \eta_r a_r & \binom{\eta_r}{2}a_r^2 & \cdots & a_r^{\eta_r} & 0 & \cdots & \cdots & 0 \\
0 & 1 & \eta_r a_r & \binom{\eta_r}{2}a_r^2 & \eta_r a_r^{\eta_r-1} & a_r^{\eta_r} & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 & \eta_r a_r & \binom{\eta_r}{2}a_r^2 & \cdots & a_r^{\eta_r}
\end{vmatrix} \tag{3}$$

It will be noted that in each row the numerical coefficients are the binomial

coefficients that occur in the expansion of $(x_1+a_ix_2)^{\eta_i}$. Consider the rows involving any particular a_i. It will be noted that in the jth of these rows the first non-zero element appears in the jth column, where $1 \leqq j \leqq n_i$, and n_i is the number of rows in a_i. The order of the determinant is

$$n + 1 = \sum_{i=1}^{r} n_i.$$

It can be considered as the eliminant of the $n+1$ forms

$$x_1^{\lambda} x_2^{\mu} (x_1 + a_i x_2)^{\eta_i}, \tag{4}$$

where $1 \leqq i \leqq r$, λ takes on all values from $n_i - 1$ to 0 inclusive for each value of i, $\lambda + \mu = n_i - 1$, and $n_i + \eta_i - 1 = n$.

3. We shall now prove:

Lemma 1. *The determinant* (3) *is a homogeneous polynomial in the quantities* $a_i (1 \leqq i \leqq r)$.

In considering the rows involving any particular a_i it will be observed that the element in the jth of these rows and the $r_j^{(i)}$ th column of the determinant, where $r_j^{(i)} \geqq j$, involves a_i to the $(r_j^{(i)} - j)$th power. The degree of any term of the expansion of (3) in a_i is therefore

$$\sum_{j=1}^{n_i} (r_j^{(i)} - j) =$$

$$\sum_{j=1}^{n_i} r_j^{(i)} - \tfrac{1}{2} n_i (n_i + 1) \tag{5}$$

and the total degree of the term is

$$\sum_{i=1}^{r} \sum_{j=1}^{n_i} (r_j^{(i)} - j) = \tfrac{1}{2}(n+1)(n+2) - \sum_{i=1}^{r} \tfrac{1}{2} n_i (n_i + 1). \tag{6}$$

Since this expression depends only upon the order of the determinant and the number of rows in each quantity a_i, the lemma is proved.

Lemma 2. *Any minor of* (3) *of order of* n_i *involving only one of the quantities* a_i, *is homogeneous in that quantity.*

As may be seen from (5), the degree of any term of such a minor is dependent only upon the number of columns of (3) represented in the minor and on the value of n_i. The truth of the lemma is therefore obvious.

4. Instead of considering the forms (4), we shall consider the equivalent set of $\sum_{i=1}^{i} n_i$ forms that result when for each particular value of i we multiply the j_ith and following forms involving a_i respectively by the terms of $(1+a_i)^{n_i - j_i}$ and add, where j_i ranges from 1 to n_i. The resulting forms are

$$x_2^{k_i} (x_1 + a_i x_2)^{n - k_i}, \text{ where } 0 \leqq k_i \leqq n_i - 1. \tag{7}$$

The eliminant of these forms is

$$
(8)\qquad
\begin{vmatrix}
1 & \binom{n}{1}a_1 & \binom{n}{2}a_1^2 & \cdots & a_1^n \\
0 & 1 & \binom{n-1}{1}a_1 & \cdots & a_1^{n-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 \cdots 0\ 1 & \binom{n-k_1}{1}a_1 & \cdots a_1^{n-k_1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 \cdots 0 & 1 & \binom{\eta_1}{1}a_1 & \cdots & a_1^{\eta_1} \\
1 & \binom{n}{1}a_2 & \binom{n}{2}a_2^2 & \cdots & a_2^n \\
0 & 1 & \binom{n-1}{1}a_2 & \cdots & a_2^{n-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 \cdots 0\ 1 & \binom{\eta_2}{1}a_2 & \cdots a_2^{\eta_2} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \binom{n}{1}a_r & \binom{n}{2}a_r^2 & \cdots & a_r^n \\
0 & 1 & \binom{n-1}{1}a_r & \cdots & a_r^{n-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 \cdots 0\ 1 & \binom{\eta_r}{1}a_r & \cdots a_r^{\eta_r}
\end{vmatrix}.
$$

This determinant is equal to (3), since it is merely the result of adding to each row of (3) certain multiples of the rows following it that contain the same a_i.

Replacing $x_1^{\lambda}x_2^{\mu}$ by y_μ where $0 \leqq \mu \leqq n$ and $\lambda+\mu=n$, in the expanded form of (7), we have a system of $n+1=\sum_{i=1}^{r} n_i$ linear forms in as many variables:

$$
(9)\qquad \sum_{i=0}^{n-k_i} \binom{n-k_i}{j} a_i^{j} y_{j+k_i},
$$

where $0 \leqq i \leqq r$, and $0 \leqq k_i \leqq n_i - 1$ for each value of i.

If the forms (7) are subjected to the transformation

$$x_1 = x_1' - a_1 x_2', \quad x_2 = x_2', \tag{10}$$

the resulting forms will be

$$x_2'^{k_i}[x_1' + (a_i - a_1)x_2']^{n-k_i},$$

where $0 \leqq k_i \leqq n_i - 1$. Hence

$$y_\mu = (x_1' - a_1 x_2')^\lambda x_2'^\mu = \sum_{k=0}^{\lambda} (-1)^k \binom{\lambda}{k} a_1^k x_1'^{\lambda-k} x_2'^{k+\mu}.$$

Let $x_1'^{\lambda-k} x_2'^{k+\mu} = y'_{k+\mu}$. The transformation induced on the y_μ''s by (10) is

$$y_\mu = \sum_{k=0}^{\lambda} (-1)^k \binom{\lambda}{k} a_1^k y'_{k+\mu}, \tag{11}$$

where $0 \leqq \mu \leqq n$, and $\lambda + \mu = n$. This is a transformation of determinant unity.

The transformation (11) reduces the forms (9) to the forms y_j' for $1 \leqq j \leqq n_1$, and

$$\sum_{j=k_i}^{n} \binom{n-k_i}{j-k_i} (a_1 - a_i)^{j-k_i} y_j',$$

where $1 < i \leqq r$ and $0 \leqq k_i \leqq n_i - 1$ for the remaining forms. The eliminant of these forms is equal to the eliminant of the forms (9) as a consequence of the following well known theorem:

"If a system of n linear forms in n variables with matrix a is subjected to a linear transformation with matrix c, the resulting system has as its matrix ac."

This eliminant is obviously equal to its $\sum_{i=2}^{r} n_i$-rowed minor that lies in the lower right hand corner. If this minor be expanded according to its n_i rows involving $(a_i - a_1)$, it is evident from (5) that each of the terms in this expansion will be divisible by $a_i - a_1$ raised to the power

$$\sum_{j=1}^{n_i} (n_1 + j) - \tfrac{1}{2} n_i (n_i + 1) = n_1 n_i.$$

If we replace a_1 in the transformation (11) by the other a_i's in turn, it is evident that the resulting eliminant, and hence the eliminant of the forms (9), is divisible by $(a_i - a_j)^{n_i n_j}$, and hence by

$$\prod_{i=1}^{r-1} \prod_{j=i+1}^{r} (a_i - a_j)^{n_i n_j}.$$

The degree of any term of this product is

$$\sum_{i=1}^{r-1} \sum_{j=i+1}^{r} n_i n_j.$$

But from (6) the degree of any term of the expansion of the determinant (3) is

$$\tfrac{1}{2}(n+1)(n+2) - \tfrac{1}{2}\sum_{i=1}^{r} n_i(n_i+1) = \tfrac{1}{2}\sum_{i=1}^{r} n_i \left(\sum_{i=1}^{r} n_i + 1 \right)$$

$$- \tfrac{1}{2}\sum_{i=1}^{r} n_i(n_i+1) = \tfrac{1}{2}\sum_{i=1}^{r} \sum_{j=1}^{r} n_i n_j + \tfrac{1}{2}\sum_{i=1}^{r} n_i - \tfrac{1}{2}\sum_{i=1}^{r} n_i(n_i+1)$$

$$= \tfrac{1}{2}\sum_{i=1}^{r} \sum_{j=1}^{r} n_i n_j - \tfrac{1}{2}\sum_{i=1}^{r} n_i^2 = \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} n_i n_j,$$

and is thus the same as that of the product referred to above. Hence (3) is equal to

$$k \prod_{i=1}^{r-1} \prod_{j=i+1}^{r} (a_i - a_j)^{n_i n_j}, \tag{12}$$

where k is a numerical multiplier to be determined in the next section. This completes the proof of Theorem I.

5. It remains to prove Theorem II, i.e. to determine k in (12). This can be done by comparing a particular term of (12) with the corresponding term of the determinant (8), which has been shown to be equal to (3). The proofs of Lemmas 1 and 2 hold true mutatis mutandis for (8).

We shall designate the n_i-rowed minors in a_i that lie along the leading diagonal of (8) by M_i, where $1 \leqq i \leqq r$; the $\sum_{j=1}^{i} n_j$-rowed minors in the upper left hand corner of (8) by D_i. D_r is evidently the determinant itself. All the terms of the expansion of M_i are of the same degree in a_i by Lemma 2. M_i is a unique minor of D_i of maximum degree in a_i by (5). Now all the terms of (8) of maximum degree in a_r arise as the product of M_r by D_{r-1}. Of these terms, those that involve a_{r-1} to a maximum degree arise as the product $M_r M_{r-1} D_{r-2}$. Of these terms, those that involve a_{r-2} to a maximum degree arise as the product $M_r M_{r-1} M_{r-2} D_{r-3}$. Continuing this process we see that $\prod_{i=1}^{r} M_i$ is a unique term in the expansion of (8) of which the coefficient is k.

6. In order to find k it is merely necessary to find the values of the determinants that result when in the minors M_i we omit the quantities a_i. For convenience we shall now assume that, if necessary, the rows in (8) have been so permuted that each $n_i \geqq n_{i+1}$. These determinants are:

$$(13)\qquad \begin{vmatrix} \binom{n}{\sum_{j=1}^{i-1} n_j} & \binom{n}{\sum_{j=1}^{i-1} n_j + 1} & \cdots & \binom{n}{\sum_{j=1}^{i-1} n_j + n_i - 1} \\ \binom{n-1}{\sum_{j=1}^{i-1} n_j - 1} & \binom{n-1}{\sum_{j=1}^{i-1} n_j} & \cdots & \binom{n-1}{\sum_{j=1}^{i-1} n_j + n_i - 2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \binom{n-n_i+1}{\sum_{j=1}^{i-1} n_j - n_i + 1} & \binom{n-n_i+1}{\sum_{j=1}^{i-1} n_j - n_i + 2} & \cdots & \binom{n-n_i+1}{\sum_{j=1}^{i-1} n_j} \end{vmatrix}$$

where $1 < i \leqq r$.

If the last row of (13) is brought into the first place, the next to the last row into the second place, etc., and then the last column is brought into the first place, the next to the last into the second place, etc., (13) is unchanged since there are as many interchanges of rows as of columns. If we now bear in mind that

$$\binom{n}{r} = \binom{n}{n-r},$$

we find that

$$\binom{n-n_i+1}{\sum_{j=1}^{i-1} n_j} = \binom{n-n_i+1}{n - \sum_{j=1}^{i} n_j + 1}.$$

If similar changes are made in all of the elements, the determinant becomes

$$(14)\qquad \begin{vmatrix} \binom{n-n_i+1}{n - \sum_{j=1}^{i} n_j + 1} & \binom{n-n_i+1}{n - \sum_{j=1}^{i} n_j + 2} & \cdots & \binom{n-n_i+1}{n - \sum_{j=1}^{i-1} n_j} \\ \binom{n-n_i+2}{n - \sum_{j=1}^{i} n_j + 1} & \binom{n-n_i+2}{n - \sum_{j=1}^{i} n_j + 2} & \cdots & \binom{n-n_i+2}{n - \sum_{j=1}^{i-1} n_j} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \binom{n}{n - \sum_{j=1}^{i} n_j + 1} & \binom{n}{n - \sum_{j=1}^{i} n_j + 2} & \cdots & \binom{n}{n - \sum_{j=1}^{i-1} n_j} \end{vmatrix}$$

The values of these determinants that result from M_1 and M_r are evidently 1. In general, (14) is a determinant of a type already discussed.[1] It has been shown that

$$\begin{vmatrix} \binom{m}{p} & \binom{m}{p+1} & \cdots & \binom{m}{p+r} \\ \binom{m+1}{p} & \binom{m+1}{p+1} & \cdots & \binom{m+1}{p+r} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{m+r}{p} & \binom{m+r}{p+1} & \cdots & \binom{m+r}{p+r} \end{vmatrix} = \frac{\binom{m+r}{r+1}}{\binom{p+r}{r+1}} \cdot \frac{\binom{m+r-1}{r+1}}{\binom{p+r-1}{r+1}} \cdots \frac{\binom{m+r-p+1}{r+1}}{\binom{r+1}{r+1}}.$$

Hence (14) is equal to

$$\frac{\binom{n}{n_i}}{\binom{n-\sum_{j=1}^{i-1} n_j}{n_i}} \cdot \frac{\binom{n-1}{n_i}}{\binom{n-\sum_{j=1}^{i-1} n_j - 1}{n_i}} \cdots \frac{\binom{\sum_{j=1}^{i} n_j}{n_i}}{\binom{n_i}{n_i}}.$$

and the numerical coefficient, k, referred to at the end of the preceding paragraph is

$$(15) \qquad \prod_{i=2}^{r-1} \frac{\binom{n}{n_i}}{\binom{n-\sum_{j=1}^{i-1} n_j}{n_i}} \cdot \frac{\binom{n-1}{n_i}}{\binom{n-\sum_{j=1}^{i-1} n_j - 1}{n_i}} \cdot \cdots \cdot \frac{\binom{\sum_{j=1}^{i} n_j}{n_i}}{\binom{n_i}{n_i}}.$$

And this is the value of k in (12).

7. We shall now determine the invariant factors of the matrix (4), and as a corollary find the rank of the system of equations (1) for the case where two of the a_i's are equal.

We shall refer to the eliminant of the forms that result when the transformation (11) is applied to the forms (9) as (8′). It has been pointed out that (8)

[1] E. Pascal in *Die Determinanten*, pages 133, 134.

is obtained from (4) by elementary transformations, and that the determinant of the transformation (11) is independent of the a_i's. Hence the greatest common divisor of all minors of a given order in (4) is the same as in (8′). It is desired to choose a minor of given order in (8′) that contains the lowest power of a_2-a_1. (There is no real loss in generality in dealing with a_2-a_1 instead of a_i-a_j. It will again be convenient so to arrange the quantities that n_1 shall be greater than or equal to n_2.) It is evident from Laplace's development in terms of minors from the first rows that any minor of (8′) that is not equal to zero that contains certain of the first n_1 rows must contain the columns with the same numbers. If any minor does not contain any rows involving powers of a_2-a_1 it cannot be divisible by a_2-a_1, since none of the other rows involves a_2. From Laplace's development of a minor of (8′) in terms of minors from the rows involving a_2-a_1 it follows that the minors will be of minimum degree in a_2-a_1 when the number of rows in a_2-a_1 is as small as possible.

Consider a determinant of order $n-l+1$ of the matrix which results when in (8′) l of the rows in a_2-a_1 are omitted ($l \leqq n_2$). Each of these determinants is equal to its $(n-n_1-l+1)$-rowed minor in the lower right hand corner as follows from Laplace's development. For the n_1-rowed minor in the upper left hand corner has the value 1, and all the other n_1-rowed minors from the first n_1 rows are zero. Now expand this minor in terms of minors of order n_2-l from the n_2-l rows involving a_2-a_1. Any minor chosen entirely from rows containing powers of a_2-a_1 will, when expanded be a homogeneous function of a_2-a_1. This can be seen from arguments essentially the same as those used in the proof of Lemma 2. The degree of any element in this minor will be N_i-M_j, where N_i is the number of the column of (8′) from which the element is taken and M_j is the number of the row in a_2-a_1. $\Sigma N_i-\Sigma M_j$ is thus the degree of any term in the expansion of this minor. To make the minor of minimum degree we must take the N's as small as possible and the M's as large as possible.

Theorem III. *Every minor of order* $n-l+1$ *is divisible by* $(a_2-a_1)^{(n_1-l)(n_2-l)}$, *and there is no higher power of* a_2-a_1 *that divides all such minors.*

We now wish to show that there is a minor of minimum degree in a_2-a_1 and of order $n-l+1$ that is different from zero (if the a_i's are all distinct) and that is divisible by exactly the power specified in the theorem. To obtain this minor we omit rows numbered n_1+1, n_1+2, $\cdots$ n_1+l, and the columns numbered n_1+n_2-l+1, n_1+n_2-l+2, $\cdots$, n_1+n_2. As was pointed out above, this minor is equal to its $(n-n_1-l+1)$-rowed minor in the lower right hand corner. In the latter, which we shall refer to as (8″), it will be noted that the (n_2-l)-rowed minor in the upper left hand corner is the minor of minimum degree in a_2-a_1 described above, and its complementary minor of order $(n-n_1-n_2+1)$ in the lower right hand corner is the corresponding minor of (8′). Expand (8″) according to Laplace's Development in terms of the (n_2-l)-rowed minors from the first (n_2-l) rows. Each of these minors is homogeneous in a_2-a_1. The term of this sum that involves the leading (n_2-l)-rowed minor involves a_2-a_1 to a lower

power than any other term provided it is not identically zero. Since this term involves a_2-a_1 to the power

$$\sum_{i=1}^{n_2-l}(n_1+i)-\sum_{j=0}^{n_2-l-1}(n_2-j)=\frac{n_2-l}{2}(2n_1+n_2-l+1)$$

$$-\frac{n_2-l}{2}(n_2+l+1)=(n_2-l)(n_1-l),$$

the expansion is divisible by $(a_2-a_1)^{(n_1-l)(n_2-l)}$. The argument given in paragraph 5 shows that the complementary minor of this leading minor is not identically zero. The leading minor itself is not zero, for the determinant of the numerical coefficients is a determinant of the type discussed in paragraph 6.

If we now replace a_2-a_1 by a_i-a_j and allow i and j to take on all possible values from 1 to r inclusive, it is evident that if the product is taken over pairs of values $i, j, (j>i)$ for which $n_i>1$, $n_j>1$, the highest common factor of all determinants of order $n-l+1$ of the matrix (3) is

$$k\prod_i\prod_j(a_i-a_j)^{(n_i-l)(n_j-l)},$$

where k is a numerical constant. From this result the invariant factors and the elementary divisors can readily be determined.

Corollary: *The rank of the system of equations (1) for the case where two of the a_i's are equal is $n+1-min(n_i, n_j)$.*

For suppose that two of the a_i's are alike, in particular, that $a_1=a_2$. It is evident from the preceding results that any minor of (3) will contain (a_2-a_1) as a factor, and hence vanish if $l<n_2\leqq n_1$. If $l=n_2$ there is a minor of order $n-l+1$, selected in the manner described in the proof of the preceding theorem, that involves a_2-a_1 to the zeroth power, and does not vanish. Hence, in general if two of the a_i's are alike, and all the other a_i's are different from each other and from these two, the rank of (3) is reduced by the lower of the two corresponding n_i's.

It is evident from (8) that, if two of the a_i's are equal, its rank is reduced by at least this much, since there are then exactly that number of pairs of rows that are identical. An extension may obviously be made to the more general case in which there are a larger number of equalities connecting the a_i's. Here, after striking out from (8) all but one row in each set of identical rows, the rank of the resulting matrix may be shown to be equal to the number of rows that it contains by a method similar to that employed above.